Influence of susceptibility - probability distributions

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If $Z \sim \ N\left(\mu, \sigma^2\right)$ then $E\left(e^z\right) = \int_{-\infty}^{+\infty} e^z \varphi\left(z\right) dz$ with $\varphi\left(z\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

$$\begin{aligned} E\left(e^z\right) &= \int_{-\infty}^{+\infty} e^z \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^z e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \end{aligned} \tag{1}$$

$$\begin{aligned} z - \frac{\left(z-\mu\right)^2}{2\sigma^2} &= \frac{1}{2}\left(\frac{-\left(z-\mu\right)^2}{\sigma^2} + 2z\right) \\ &= \frac{1}{2}\left(\frac{-\left(z-\mu\right)^2}{\sigma^2} + 2\left(z-\mu\right) - \sigma^2\right) + \mu + \frac{\sigma^2}{2} \\ &= \frac{-1}{2}\left(\frac{z-\mu}{\sigma} - \sigma\right)^2 + \mu + \frac{\sigma^2}{2} \end{aligned}$$

$$\begin{aligned} (1) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-\sigma\right)^2} e^{\mu+\frac{\sigma^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-\sigma\right)^2} dz \end{aligned} \tag{2}$$

Put $w = \frac{z-\mu}{\sigma} - \sigma \Rightarrow dw = \frac{1}{\sigma} dz \Rightarrow dz = \sigma dw$.

$$\begin{aligned} (2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} \sigma dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu+\frac{\sigma^2}{2}} \sqrt{2\pi} \\ &= e^{\mu+\frac{\sigma^2}{2}} \end{aligned}$$

$Var\left(e^{z}\right)=E\left[e^{2z}\right]-\left(E\left[e^{z}\right]\right)^{2}$

$$\begin{aligned}E\left[e^{2z}\right]&=\int_{-\infty}^{+\infty}e^{2z}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(z-\mu)^{2}}{2\sigma^{2}}}dz\\&=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}e^{2z-\frac{(z-\mu)^{2}}{2\sigma^{2}}}dz\end{aligned}\tag{3}$$

$$\begin{aligned}2z-\frac{(z-\mu)^{2}}{2\sigma^{2}}&=\frac{1}{2}\left(\frac{(z-\mu)^{2}}{\sigma^{2}}+4z\right)\\&=\frac{1}{2}\left(-\frac{(z-\mu)^{2}}{\sigma^{2}}+4\left(z-\mu\right)-4\sigma^{2}\right)+2\mu+2\sigma^{2}\\&=\frac{-1}{2}\left(\frac{z-\mu}{\sigma}-2\sigma\right)^{2}+2\mu+2\sigma^{2}\end{aligned}$$

$$(3)=\frac{1}{\sqrt{2\pi}\sigma}e^{2\mu+2\sigma^{2}}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-2\sigma\right)^{2}}dz\tag{4}$$

Put $w=\frac{z-\mu}{\sigma}-2\sigma\Rightarrow dw=\frac{dz}{\sigma}\Rightarrow dz=\sigma dw$.

$$\begin{aligned}(4)&=\frac{1}{\sqrt{2\pi}\sigma}e^{2\left(\mu+\sigma^{2}\right)}\int_{-\infty}^{+\infty}e^{\frac{-w^{2}}{2}}\sigma dw\\&=\frac{1}{\sqrt{2\pi}}e^{2\left(\mu+\sigma^{2}\right)}\int_{-\infty}^{+\infty}e^{\frac{-w^{2}}{2}}dw\\&=\frac{1}{\sqrt{2\pi}}e^{2\left(\mu+\sigma^{2}\right)}\sqrt{2\pi}\\&=e^{2\left(\mu+\sigma^{2}\right)}\end{aligned}$$

$$\begin{aligned}Var\left(e^{z}\right)&=e^{2\left(\mu+\sigma^{2}\right)}-e^{2\mu+\sigma^{2}}\\&=\left(e^{\sigma^{2}}-1\right)e^{2\mu+\sigma^{2}}\end{aligned}$$

$E\left(e^{z}\right)=1\Leftrightarrow\mu=-\frac{\sigma^{2}}{2}$
$b_0\sim\ N\left(\mu_0,\sigma_0^{2}\right)$ and $b_1\sim\ N\left(\mu_1,\sigma_0^{1}\right)$
The following conditions have to be fulfilled:
$E\left(e^{b_0+b_1}\right)=e^{\mu_0+\mu_1+\frac{\left(\sigma_0^{2}+\sigma_1^{2}\right)}{2}}=1$
$E\left(e^{b_0+b_2}\right)=e^{\mu_0+\mu_2+\frac{\left(\sigma_0^{2}+\sigma_2^{2}\right)}{2}}=1$
This is a.o. the case if $\mu_0=-\frac{\sigma_0^{2}}{2}$, $\mu_1=-\frac{\sigma_1^{2}}{2}$ and $\mu_2=-\frac{\sigma_2^{2}}{2}$